Gomory Cut in Integer Programming

(A presentation on Mathematical Programming)

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Solution of Integer Programming Problem using Gomory Cut

This method is also called the **Cutting Plane Method**. The following steps are employed. (The variables are assumed to be non-negative and the problem is maximization type.)

- First solve the corresponding LP relaxation problem, that is, solve the given problem for real values by using the simplex method.
- As obtaining the integer optimal solution at this stage is shall mean no application of the method we are going to discuss, we shall assume, with no loss in generality, that we have not obtained the integer solution at this stage.

- Construction of Gomory Cut

Choose a constraint in the optimal simplex table which corresponds to a non-integer value of a variable. Let it be $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$. Let $[b]$ denote the integer part of b and $\{b\}$ denote the fractional part of b. Note that $1 > \{b\} \geq 0$. To obtain the Gomory cut, we write b as $[b] + \{b\}$ and do the same for every coefficient in the constraint. Take all the integral parts of the equation to the left and the fractional parts to the right. The constraint would, then, look like

$$[a_1]x_1 + [a_2]x_2 + \cdots + [a_n]x_n - [b] = -\{a_1\}x_1 - \cdots + \{b\}.$$

We are seeking for the integer solution of the problem. For obtaining one, we shall have all the x's in the above constraint as integers. If so, then the left-hand side is an integer and so is the right-hand side. But due to the presence of fractional parts, it follows that the equation does not represent a positive integer and hence in such a case,

$$[a_1]x_1 + [a_2]x_2 + \cdots + [a_n]x_n - [b] = -\{a_1\}x_1 - \cdots + \{b\} \leq 0.$$

Now, we consider

$$-\{a_1\}x_1 - \cdots - \{a_n\}x_n + \{b\} \leq 0$$

as a secondary constraint for the problem under consideration. This is what we refer to as **Gomory Cut** or **Gomory constraint**.

- We now include the Gomory cut as a constraint in the optimal simplex table. To do so, we introduce a slack variable, say S, and rewrite the cut as

  $$-\{a_1\}x_1 - \cdots - \{a_n\}x_n + S = -\{b\}.$$

  Inclusion of this constraint in the optimal table makes the solution optimal, but infeasible.

- To “re-obtain” the feasibility, we apply the **Dual - Simplex Algorithm**. For this, we fix the new row we have introduced. From the row that contains the coefficients of cost, we take the ratio of a cost element and the corresponding element in the fixed row, provided that the row element is negative for this purpose. The column corresponding to the maximum ratio is then fixed. Pivoting is, then, done as usual and we check for integer optimality. If we do not obtain it, we repeat the entire procedure again.

Illustration

Consider the problem

$$\begin{gathered}\max\ z = x + y \\ \text{subjected to} \\ 2x + 5y \leq 16 \\ 6x + 5y \leq 30 \\ x,\ y \in \mathbb{Z}_{\geq 0}.\end{gathered}$$

The optimal table for the LP relaxation problem (the one with same constraints, except, $x,\ y \geq 0$) is shown below.

v $\rightarrow$	x	y	s_1	s_2	$\vec{b}$
b.v. $\downarrow$	0	0	1/20	3/20	53/10
y	0	1	3/10	-1/10	9/5
x	1	0	-1/4	1/4	7/2

Here, $z_{\max} = \frac{53}{10}$ at $\left(\frac{7}{2}, \frac{9}{5}\right)$. This is not an integer optimal solution. To use the Gomory cut for the solution, we select the first row as the fractional part of 9/5 is greater than that of 7/2. The constraint is

$$y + \frac{3}{10}\, s_1 - \frac{1}{10}\, s_2 = \frac{9}{5}$$

which can be rearranged as

$$y - s_2 - 1 = -\frac{3}{10}\, s_1 - \frac{9}{10}\, s_2 + \frac{4}{5}$$

and as discussed before, we get the Gomory cut as

$$-\frac{3}{10}\, s_1 - \frac{9}{10}\, s_2 + \frac{4}{5} \leq 0.$$

To introduce this Gomory cut in the above optimal table, we introduce a slack variable s_3 and re-write the cut as

$$-\frac{3}{10}\, s_1 - \frac{9}{10}\, s_2 + s_3 = -\frac{4}{5}.$$

When we introduce this equation as a constraint in the previous optimal table we get the following table. The ratio for the dual-simplex algorithm is also calculated there.

v $\rightarrow$	x	y	s_1	s_2	s_3	$\vec{b}$
b.v. $\downarrow$	0	0	1/20	3/20	0	53/10
y	0	1	3/10	-1/10	0	9/5
x	1	0	-1/4	1/4	0	7/2
s_3	0	0	-3/10	-9/10	1	-4/5
Ratio $\rightarrow$	-	-	-1/6	-1/6	-	$\times$

We fix the last row, as it represents the Gomory cut. Now, we take ratios of the element of cost row and the corresponding element in the last row, if the element of the last row is negative. Doing so, we can select the column corresponding to s_1 as the pivot column. Applying elementary row operations we get the above tableau as shown below.

v $\rightarrow$	x	y	s_1	s_2	s_3	$\vec{b}$
b.v. $\downarrow$	0	0	0	0	1/6	31/6
y	0	1	0	-1	1	1
x	1	0	0	1	-5/6	25/6
s_1	0	0	1	3	-10/3	8/3

This isn't an integer optimal solution, either. So, selecting the second row as a source row, and, proceeding with the Gomory cut and dual-simplex algorithm in a similar fashion, we get the following optimal table.

v $\rightarrow$	x	y	s_1	s_2	s_3	s_4	$\vec{b}$
b.v. $\downarrow$	0	0	0	0	0	1/6	5
y	0	1	0	-1	0	0	0
x	1	0	0	1	0	-5/6	5
s_1	0	0	1	3	0	-10/3	6
s_3	0	0	0	0	1	-1	1

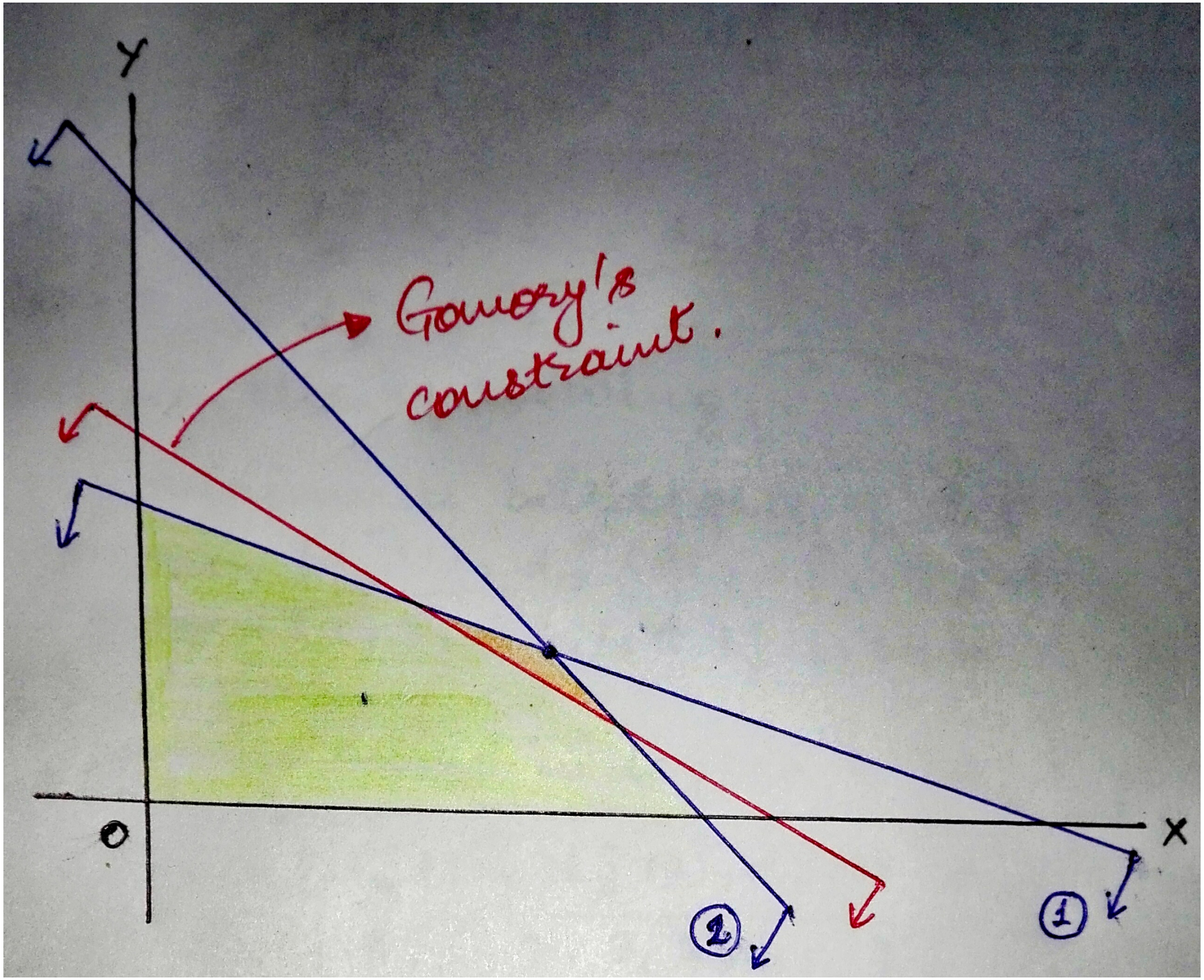
This shows the integer optimality and hence, we stop the application of the algorithm here. Thus, $z_{\max} = 5$ at (5, 0).

Geometrical interpretation of the method

The first Gomory cut of the problem in the illustration is

$$\frac{3}{10}\, s_1 + \frac{9}{10}\, s_2 \geq \frac{4}{5}.$$

But for the geometrical interpretation, we need this in terms of x and y. We know that $2x + 5y + s_1 = 16$ and $6x + 5y + s_2 = 30$. Using these, we get the Gomory cut as $x + y \leq \frac{31}{6}$. Taking this as a constraint, and plotting it geometrically, we get the feasible region as shown below. It has divided the feasible region into two parts, hence, "reducing" it. We can proceed with second Gomory cut similarly. As it cuts off a part of the feasible region by introducing a new constraint, the name "cutting plane method" is justified.


Y
Gomory's constraint.
O
X
2
1